

Further Differentiation – Section Test

(Answers)

1. (i) $y = x^3 + 6x^2 + 9x$

$$\frac{dy}{dx} = 3x^2 + 12x + 9$$

(ii) $\frac{dy}{dx} = 0$

$$3x^2 + 12x + 9 = 0$$

$$x^2 + 4x + 3 = 0$$

$$(x+1)(x+3) = 0$$

$$x = -1 \text{ or } x = -3$$

When $x = -1$, $y = (-1)^3 + 6(-1)^2 + 9 \times -1 = -1 + 6 - 9 = -4$

When $x = -3$, $y = (-3)^3 + 6(-3)^2 + 9 \times -3 = -27 + 54 - 27 = 0$

The stationary points are $(-1, -4)$ and $(-3, 0)$

(iii)

x	$x < -3$	$x = -3$	$-3 < x < -1$	$x = -1$	$x > -1$
$\frac{dy}{dx}$	+ve 	0 	-ve 	0 	+ve 

The point $(-3, 0)$ is a maximum point.

The point $(-1, -4)$ is a minimum point.

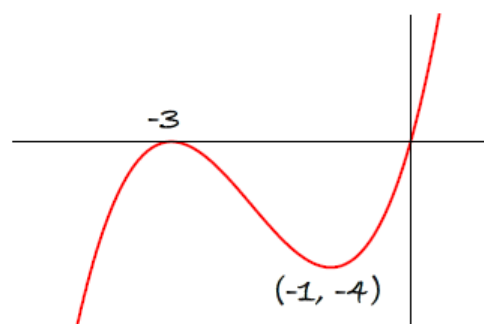
(iv) $y = x^3 + 6x^2 + 9x$

$$= x(x^2 + 6x + 9)$$

$$= x(x+3)^2$$

The graph cuts the x -axis at $x = 0$ and $x = -3$ (repeated).

The graph cuts the y -axis at $y = 0$.



2. (i) $y = 2x + x^2 - 4x^3$

$$\frac{dy}{dx} = 2 + 2x - 12x^2$$

At stationary points, $\frac{dy}{dx} = 0$

$$2 + 2x - 12x^2 = 0$$

$$1 + x - 6x^2 = 0$$

$$6x^2 - x - 1 = 0$$

$$(3x+1)(2x-1) = 0$$

$$x = -\frac{1}{3} \text{ or } x = \frac{1}{2}$$

When $x = -\frac{1}{3}$, $y = 2(-\frac{1}{3}) + (-\frac{1}{3})^2 - 4(-\frac{1}{3})^3 = -\frac{2}{3} + \frac{1}{9} + \frac{4}{27} = \frac{-18+3+4}{27} = -\frac{11}{27}$

When $x = \frac{1}{2}$, $y = 2(\frac{1}{2}) + (\frac{1}{2})^2 - 4(\frac{1}{2})^3 = 1 + \frac{1}{4} - \frac{1}{2} = \frac{3}{4}$

The stationary points are $(-\frac{1}{3}, -\frac{11}{27})$ and $(\frac{1}{2}, \frac{3}{4})$.

x	$x < -\frac{1}{3}$	$x = -\frac{1}{3}$	$-\frac{1}{3} < x < \frac{1}{2}$	$x = \frac{1}{2}$	$x > \frac{1}{2}$
$\frac{dy}{dx}$	-ve 	0 	+ve 	0 	-ve 

$(-\frac{1}{3}, -\frac{11}{27})$ is a minimum point.

$(\frac{1}{2}, \frac{3}{4})$ is a maximum point.

(ii) $y = 2x + x^2 - 4x^3$

$$= x(2 + x - 4x^2)$$

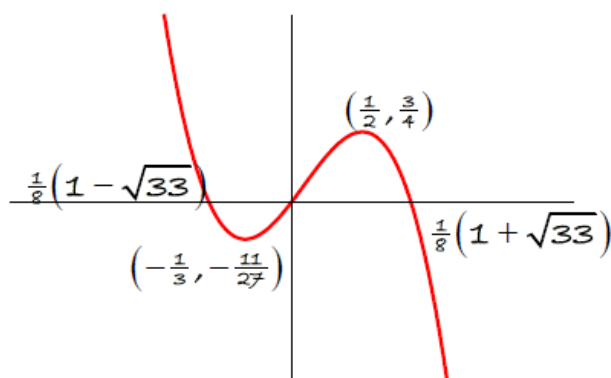
$$= -x(4x^2 - x - 2)$$

The curve cuts the x-axis at $x = 0$ and at the points satisfying

$$4x^2 - x - 2 = 0.$$

For this quadratic equation, $a = 4, b = -1, c = -2$

using the quadratic formula, $x = \frac{1 \pm \sqrt{1 - 4 \times 4 \times -2}}{8} = \frac{1 \pm \sqrt{33}}{8}$



3. $y = x^3 - 3x^2 + 6$

$$\frac{dy}{dx} = 3x^2 - 6x$$

At stationary points, $\frac{dy}{dx} = 0$

$$3x^2 - 6x = 0$$

$$3x(x - 2) = 0$$

$$x = 0 \text{ or } x = 2$$

When $x = 0$, $y = 6$

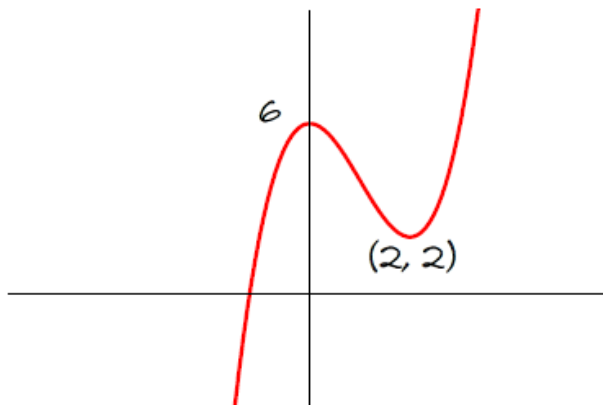
When $x = 2$, $y = 2^3 - 3 \times 2^2 + 6 = 8 - 12 + 6 = 2$

The stationary points are $(0, 6)$ and $(2, 2)$.

x	$x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$\frac{dy}{dx}$	+ve /	0 —	-ve \ 	0 —	+ve /

$(0, 6)$ is a maximum point.

$(2, 2)$ is a minimum point.



5. $y = x^4 - 2x^3$

$$\frac{dy}{dx} = 4x^3 - 6x^2$$

At stationary points, $4x^3 - 6x^2 = 0$

$$x^2(2x - 3) = 0$$

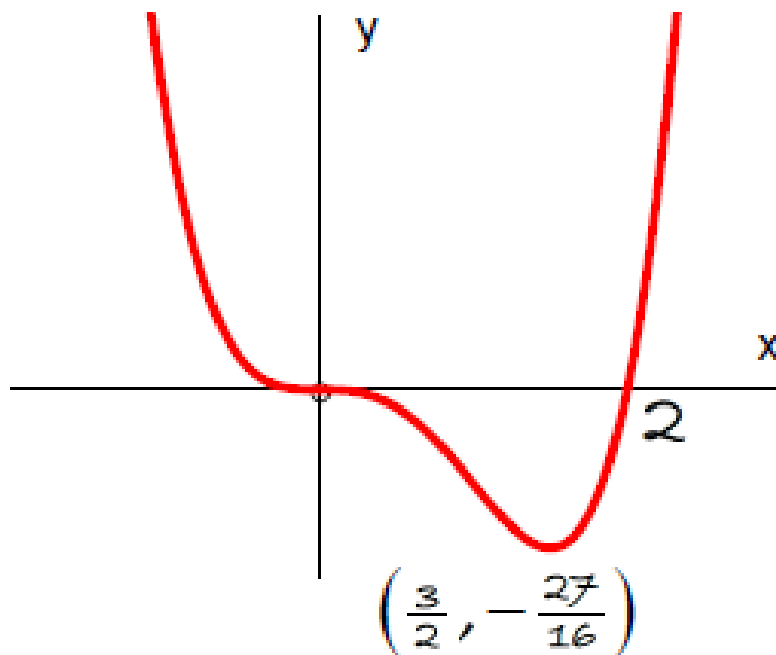
$$x = 0 \text{ or } x = \frac{3}{2}$$

When $x = 0$, $y = 0$

$$\text{When } x = \frac{3}{2}, y = \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 = \frac{81}{16} - \frac{27}{4} = -\frac{27}{16}$$

x	$x < 0$	$x = 0$	$0 < x < \frac{3}{2}$	$x = \frac{3}{2}$	$x > \frac{3}{2}$
$\frac{dy}{dx}$	-ve ↘	0 —	-ve ↘	0 —	+ve ↗

So $(0, 0)$ is a point of inflection, and $\left(\frac{3}{2}, -\frac{27}{16}\right)$ is a minimum point.



7. (i) $y = x^3 + ax^2 + bx + c$

The graph passes through the point (1, 1)

so $1 = 1 + a + b + c$

$$a + b + c = 0$$

(ii) $\frac{dy}{dx} = 3x^2 + 2ax + b$

Stationary points are when $3x^2 + 2ax + b = 0$

There is a stationary point when $x = -1$, so $3(-1)^2 + 2a \times -1 + b = 0$

$$3 - 2a + b = 0$$

$$2a - b = 3$$

There is a stationary point when $x = 3$, so $3 \times 3^2 + 2a \times 3 + b = 0$

$$27 + 6a + b = 0$$

$$6a + b = -27$$

(iii) $a + b + c = 0$ (1)

$2a - b = 3$ (2)

$6a + b = -27$ (3)

Adding (2) and (3):

$$8a = -24 \Rightarrow a = -3$$

Substituting into (2) gives:

$$b = 2a - 3 = -6 - 3 = -9$$

Substituting into (1) gives:

$$c = -a - b = 9 + 3 = 12$$

$$a = -3, b = -9, c = 12$$

$$8. \frac{dy}{dx} = 3x^2 - 10x + 3$$

At a stationary point $\frac{dy}{dx} = 0$

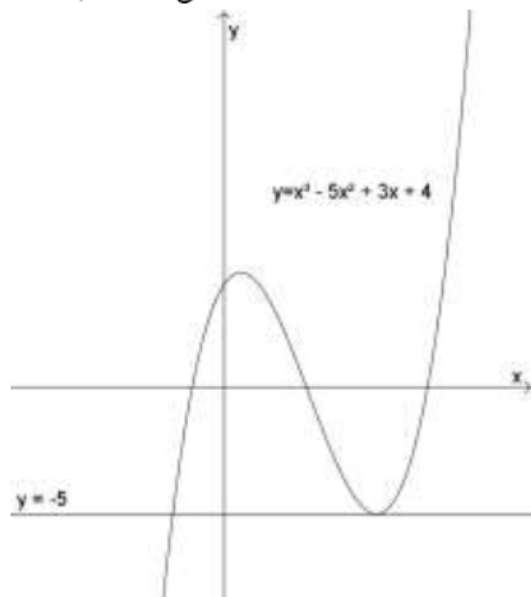
$$3x^2 - 10x + 3 = 0$$

$$(3x - 1)(x - 3) = 0$$

$x = 1/3$ or 3 .

The minimum point is when $x = 3$ (from the sketch)

At this point $y = x^3 - 5x^2 + 3x + 4 = 27 - 45 + 9 + 4 = -5$



The line $y = -5$ crosses the curve again. To the left of this crossing point, y takes values below those at the minimum stationary point.

$$x^3 - 5x^2 + 3x + 4 = -5$$

$$x^3 - 5x^2 + 3x + 9 = 0$$

$x = 3$ is a root so $(x - 3)$ is a factor.

$$x^3 - 5x^2 + 3x + 9 = 0$$

$$(x - 3)(x^2 - 2x - 3) = 0$$

$$(x - 3)(x - 3)(x + 1) = 0$$

y takes values below that at the minimum stationary point for $x < -1$.

$$9. \frac{dy}{dx} = 60x^2 - 114x + 54$$

At a stationary point $\frac{dy}{dx} = 0$

$$60x^2 - 114x + 54 = 0$$

$$10x^2 - 19x + 9 = 0$$

$$(10x - 9)(x - 1) = 0$$

The equation has two roots so there are two stationary points.

10. (i)

$$y = 10 - 3x - x^2$$

$$\frac{dy}{dx} = -3 - 2x$$

$$\frac{d^2y}{dx^2} = -2$$

(ii)

$$y = 3x(x^2 - 2x) = 3x^3 - 6x^2$$

$$\frac{dy}{dx} = 9x^2 - 12x$$

$$\frac{d^2y}{dx^2} = 18x - 12$$

(iii)

$$y = (2x + 5)(x^2 - 3x) = 2x^3 - x^2 - 15x$$

$$\frac{dy}{dx} = 6x^2 - 2x - 15$$

$$\frac{d^2y}{dx^2} = 12x - 2$$

11. (i)

$$y = -2x^3 + 6x^2 + x - 7$$

$$\frac{dy}{dx} = -6x^2 + 12x + 1$$

$$\frac{d^2y}{dx^2} = -12x + 12$$

(ii)

$$\text{at } x=1, \frac{dy}{dx} = -6(1)^2 + 12(1) + 1 = 7$$

$$\text{at } x=3, \frac{dy}{dx} = -6(3)^2 + 12(3) + 1 = -17$$

$$\text{at } x=-1, \frac{dy}{dx} = -6(-1)^2 + 12(-1) + 1 = -17$$

(iii)

$$\text{at } x=1, \frac{d^2y}{dx^2} = -12(1) + 12 = 0$$

$$\text{at } x=3, \frac{d^2y}{dx^2} = -12(3) + 12 = -24$$

$$\text{at } x=-1, \frac{d^2y}{dx^2} = -12(-1) + 12 = 24$$