PURE 2

MOCK PAPER 2

JANUARY 2019

WORKED SOLUTIONS

1 a)

$$\left.\begin{aligned} \cos\theta &\approx 1-\tfrac{1}{2}\theta^2 \\ \sin\theta &\approx \theta \\ \tan\theta &\approx 2\theta \end{aligned}\right\}$$ when θ is small

(ie close to zero)

$$\cos\theta - \sin\left(\tfrac{1}{2}\theta\right) + 2\tan\theta = \frac{11}{10}$$

$$1-\tfrac{1}{2}\theta^2 - \tfrac{1}{2}\theta + 2\theta \approx \frac{11}{10}$$

$$-\tfrac{1}{2}\theta^2 + \tfrac{3}{2}\theta + 1 \approx \frac{11}{10}$$

$$-5\theta^2 + 15\theta + 10 \approx 11$$

$$-5\theta^2 + 15\theta - 1 \approx 0$$

$$5\theta^2 - 15\theta + 1 \approx 0$$

b) $\theta = 0.068$ is valid because θ is small

$\theta = 2.932$ is NOT valid because θ is large

(2)

$x = 6t + 1 \qquad y = 5 - \frac{4}{3t} \qquad t \neq 0$

$\downarrow$

$\frac{x-1}{6} = t$

$$y = 5 - \frac{4}{3\left(\frac{x-1}{6}\right)}$$

$$y = 5 - \frac{4}{\left(\frac{x-1}{2}\right)}$$

$$y = 5 - \frac{8}{x-1}$$

$$y = \frac{5(x-1)}{(x-1)} - \frac{8}{(x-1)}$$

$$y = \frac{5(x-1) - 8}{(x-1)}$$

$$y = \frac{5x - 5 - 8}{x-1}$$

$$y = \frac{5x - 13}{x-1} \qquad x \neq 1$$

$a = 5 \quad b = -13 \quad k = 1$

3

a) $\frac{dy}{dx} = 0$ when $x = 3$

$$y = x^2 + kx + 14 - 8(x-5)^{-1}$$

$$\frac{dy}{dx} = 2x + k + 8(x-5)^{-2}$$

$$0 = 6 + k + 8(3-5)^{-2}$$

$$0 = 6 + k + 2$$

$$k = -8$$

b) inflection when $\frac{d^2y}{dx^2} = 0$

max when $\frac{d^2y}{dx^2} < 0$

min when $\frac{d^2y}{dx^2} > 0$

$$\frac{dy}{dx} = 2x + k + 8(x-5)^{-2}$$

$$\frac{d^2y}{dx^2} = 2 - 16(x-5)^{-3}$$

when $x = 3$ $\quad \frac{d^2y}{dx^2} = 4 \quad \therefore$ mim point

③ A point of inflection is a point at which $f''(x)$ changes sign.

c) at $x = 7$

$$\frac{d^2y}{dx} = 0$$ $\therefore$ possible inflection point

check eithe side

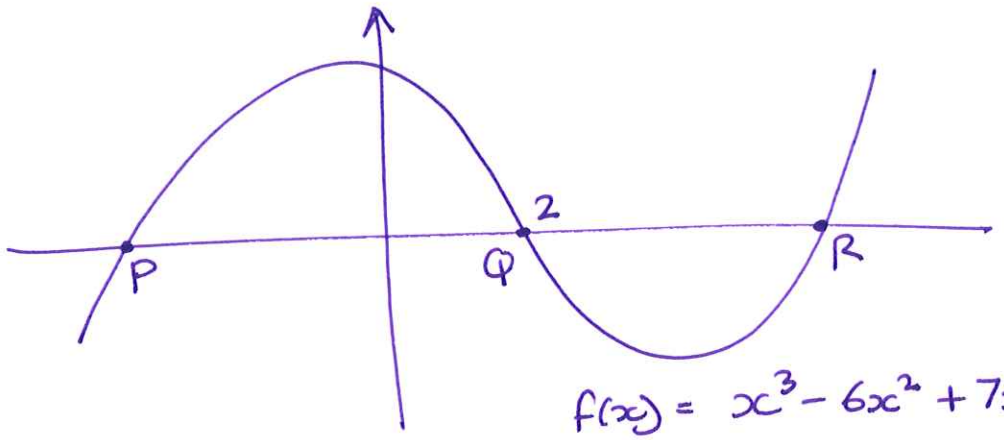
$x = 6.9$ $\quad \frac{d^2y}{dx^2} = -0.33$

$x = 7.1$ $\quad \frac{d^2y}{dx} = 0.27$

$\therefore$ there is a point of inflection at $x=7$

4

a)



$$f(x) = x^3 - 6x^2 + 7x + 2$$

$$f(x) = (x+P)(x-Q)(x-R)$$

$$(x-2)$$

$$\begin{array}{r} x^2 - 4x - 1 \\ (x-2)\overline{\big)\, x^3 - 6x^2 + 7x + 2} \\ -\quad \underline{x^3 - 2x^2} \qquad\qquad \\ -4x^2 + 7x \qquad \\ -\quad \underline{-4x^2 + 8x} \qquad \\ -x + 2 \\ -\quad \underline{-x + 2} \\ 0 \end{array}$$

$$(x-2)(x^2 - 4x - 1)$$

2 factors

4

b)

$x^2 - 4x - 1 = 0$

$(x-2)^2 - 4 - 1 = 0$ or use quadratic formula

$(x-2)^2 = 5$

$x - 2 = \pm\sqrt{5}$

$x = 2 \pm \sqrt{5}$

c)

$\sin^3\theta - 6\sin^2\theta + 7\sin\theta + 2 = 0$

$x^3 - 6x^2 + 7x + 2 = 0$

solutions are (get your calculator to do this)

i) 2

ii) $2 + \sqrt{5}$

iii) $2 - \sqrt{5}$

$\sin\theta = 2$ no solutions

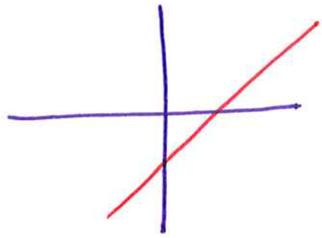
$\sin\theta = 2 + \sqrt{5}$ no solutions

$\sin\theta = 2 - \sqrt{5}$ 14 solution in interval of $\underbrace{-\pi \leq \theta}_{\text{2 solutions}} \leq \underbrace{12\pi}_{\substack{\text{6 cycles} \\ \text{2 solutions per cycle}}}$

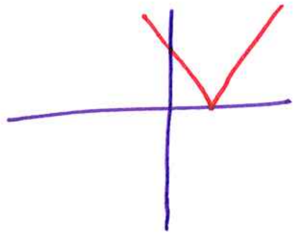
14 solutions

5

a) $f(x) = 3x - 5$

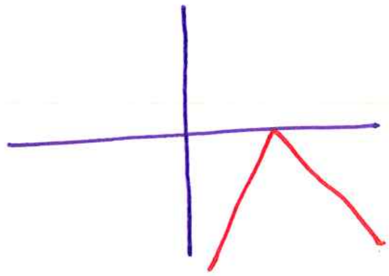


$f(x) = |3x - 5|$



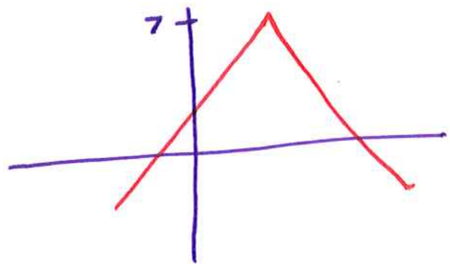
reflection in x axis

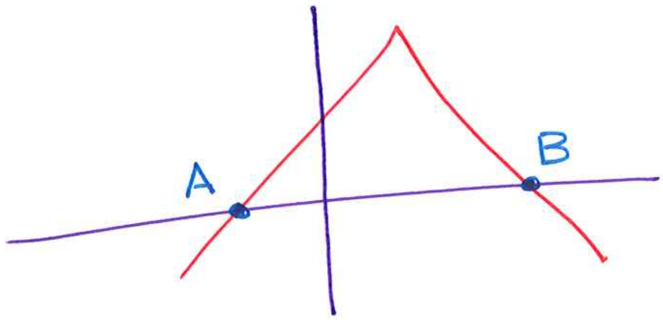
$f(x) = -|3x - 5|$



up 7

$f(x) = 7 - |3x - 5|$





A

$7 - (-3x + 5) = 0$

$7 + 3x - 5 = 0$

$3x = -2$

$x = \frac{-2}{3}$

B

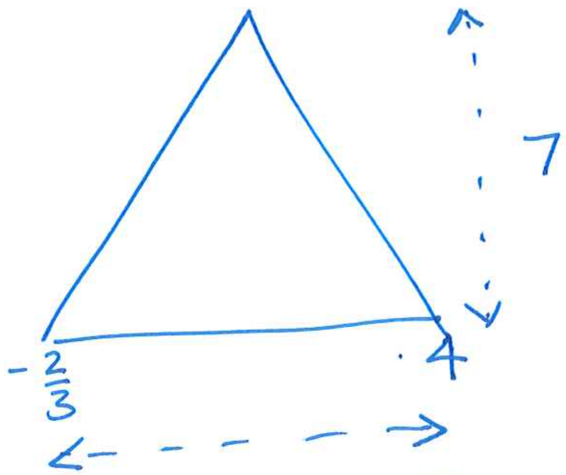
$7 - (3x - 5) = 0$

$7 - 3x + 5 = 0$

$-3x = -12$

$x = +4$

a) conti.

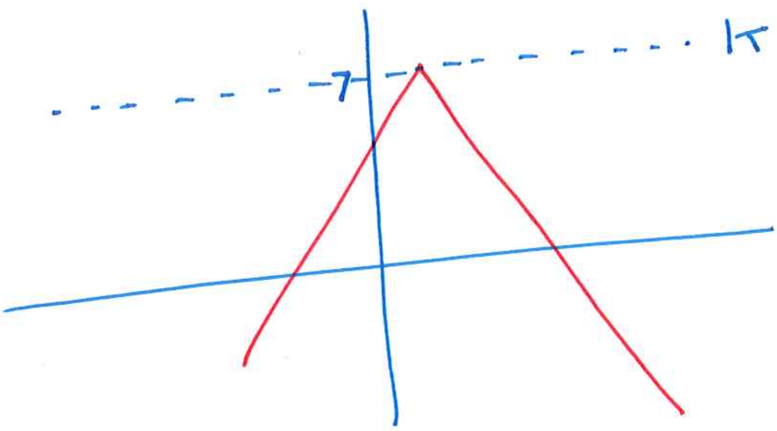


$$4 + \frac{2}{3} = \frac{14}{3}$$

$$\text{Area} = \frac{1}{2} \times 7 \times \frac{14}{3} = \frac{96}{6} = \underline{\underline{\frac{49}{3}}}$$

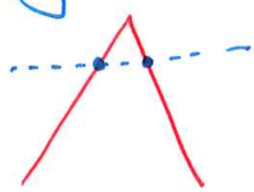
b)

$$7 - |3x - 5| = k$$



$k < 7$ to intersect in two places

$\therefore$ having 2 roots.



⑥

a)

$$(2 + kx)^{-4}$$

$$2^{-4}\left(1 + \frac{kx}{2}\right)^{-4}$$

$$n = -4$$
$$x = \frac{kx}{2}$$

$$2^{-4}\left[1 + (-4)\left(\frac{kx}{2}\right) + \frac{(-4)(-5)}{2}\left(\frac{kx}{2}\right)^2\right]$$

$$= \frac{1}{16}\left[1 - 2kx + 10\,\frac{k^2x^2}{4}\right]$$

$$= \frac{1}{16} - \frac{1}{8}kx + \frac{10}{64}k^2x^2$$

$$= \frac{1}{16} - \frac{kx}{8} + \frac{5}{32}k^2x^2$$

compare x^2

$$\frac{125}{32} = \frac{5}{32}k^2$$

$k = \pm 5$ k must be positive $\therefore k = 5$

Hence $A = -\frac{5}{8}$

(6)

b) Valid when $|x| < 1$

$$\left| \frac{1+x}{2} \right| < 1$$

$$\left| \frac{5x}{2} \right| < 1$$

$$|5x| < 2$$

$$|x| < \frac{2}{5}$$

⑦

a) $f(-1.5) = 2.943536507...$

$f(-1) = -0.367879441 2....$

sign change $\therefore$ root between -1.5 and -1

b) i) $x_3 = -1.0428$

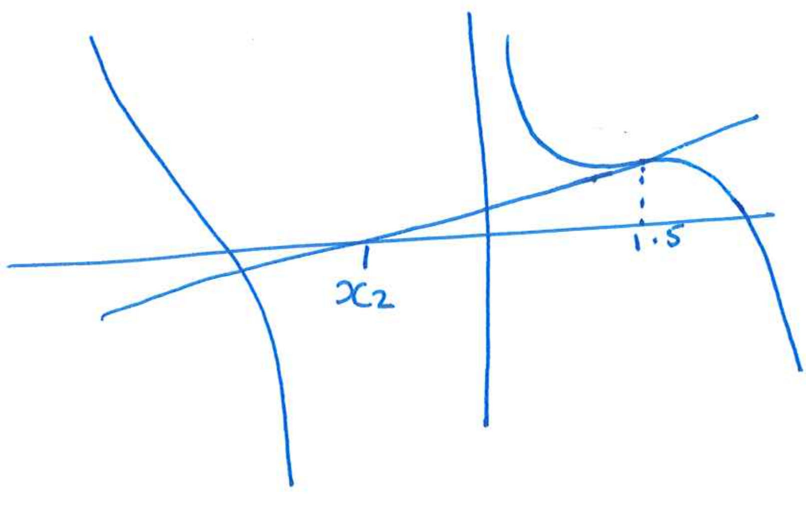
ii) $\alpha = -1.06$ (2dp)

c) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= 3 - \left(\frac{-1.4189}{-8.3078}\right)$$

$$= 2.829208695 = 2.83 \text{ (2dp)}$$

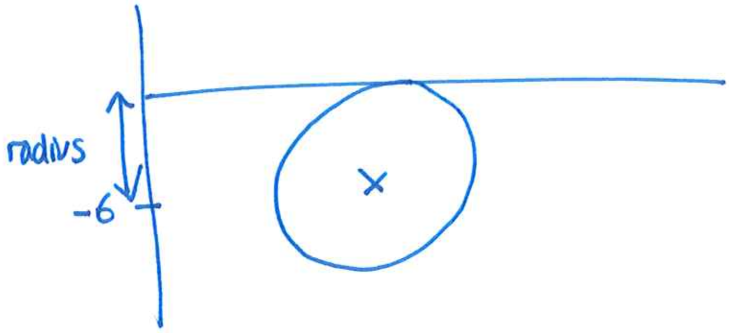
d)



x_2 nowhere near root β

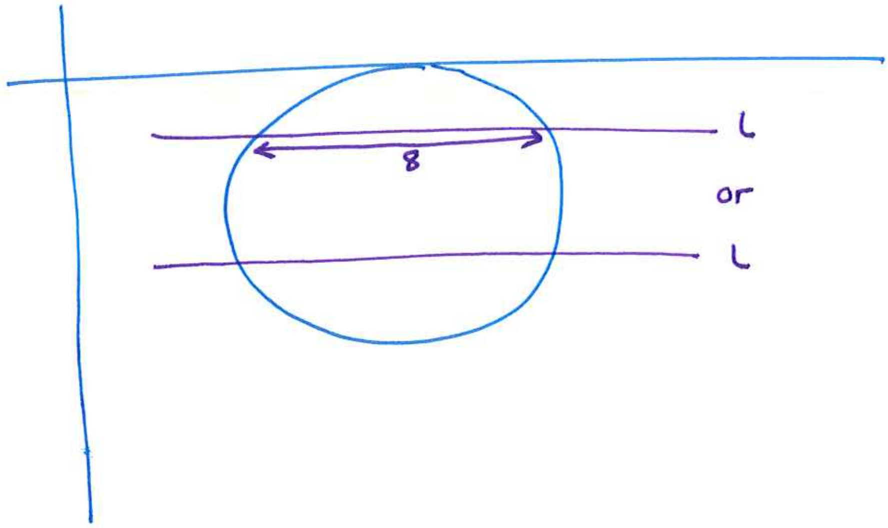
8

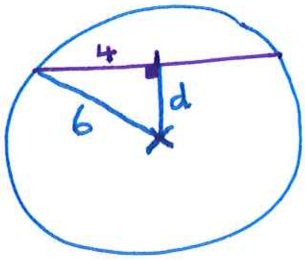
a)

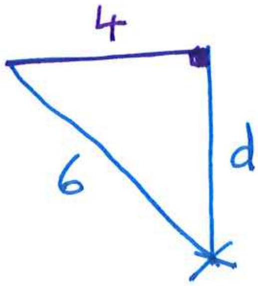


$$(x-9)^2 + (y+6)^2 = 36$$

b)



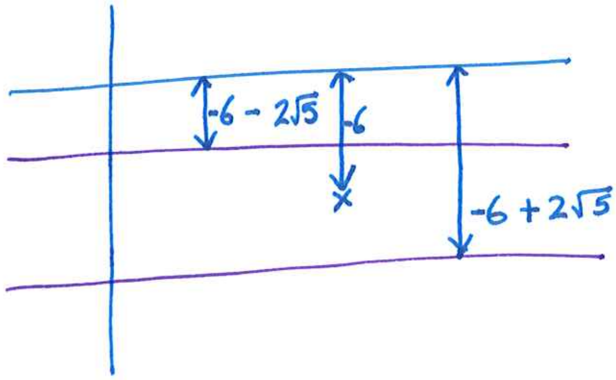




$$d^2 + 4^2 = 6^2$$

$$d^2 = 20$$

$$d = 2\sqrt{5}$$



$$y = -6 + 2\sqrt{5}$$

$$y = -6 - 2\sqrt{5}$$

9

a)

$$y = ab^t$$

$$\log y = \log ab^t$$

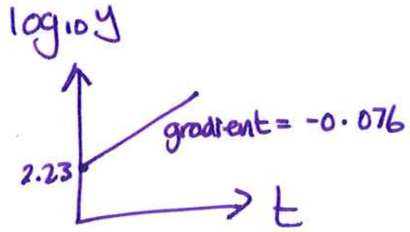
$$\log y = \log b^t + \log a$$

$$\log y = t \log b + \log a$$

$$\log y = t \log b + c \qquad \text{where } c = \log a$$

b)

$$\log_{10} y = \underbrace{t \log_{10} b}_{\text{gradient}} + \underbrace{c}_{\text{y intercept}}$$



$$\log_{10} b = -0.076$$

$$10^{-0.076} = b$$

$$b = 0.84$$

$$c = \log a$$

$$2.23 = \log a$$

$$10^{2.23} = a$$

$$a = 170$$

9 c) $y = (170)(0.84)^t$

"170" milligrams of antibiotic initially given to the patient

"0.84" after given the antibiotic reduces by 16% per hour (approx)

9 d)

$$y = ab^t$$

$a = 170$

$b = 0.84$

$$y = (170)(0.84)^t$$

$$y = 30$$

$$30 = (170)(0.84)^t$$

$$\frac{30}{170} = 0.84^t$$

$$\log \frac{30}{170} = \log 0.84^t$$

$$\log \frac{30}{170} = t \log 0.84$$

$$t = \frac{\log\left(\frac{30}{170}\right)}{\log(0.84)}$$

$t = 9.9$ hours

e) $t = 9.9$ is outside data recorded (first 5 hours) therefore may not be reliability.

(10)

$t=0 \quad A=9$

$t=6 \quad A=56.25$

a) $$\frac{dA}{dt} = k\sqrt{A}$$

$$\int \frac{1}{\sqrt{A}}\, dA = \int k \, dt$$

$$\int A^{-1/2}\, dA = \int k \, dt$$

$$\frac{A^{1/2}}{1/2} + c = kt + c$$

$$2\sqrt{A} = kt + c$$

$t=0 \quad A=9$

$$2\sqrt{9} = c$$

$$c = 6$$

$t=6 \quad A=56.25$

$$2\sqrt{A} = kt + 6$$

$$2\sqrt{56.25} = 6k + 6$$

$$k = \frac{3}{2}$$

10

a) conti

$$2\sqrt{A} = \frac{3}{2}t + 6$$

$$4\sqrt{A} = 3t + 12$$

$$\sqrt{A} = \frac{3t}{4} + 3$$

$$A = \left(\frac{3t}{4} + 3\right)^2$$

b)

$t = 12$	$A = 144$	compared to 143.78
$t = 18$	$A = 272.25$	271.19
$t = 24$	$A = 441$	334.81
$t = 30$	$A = 6502.5$	337.33

works well for $t=12$ and $t=18$

but overestimates when $t = 24$ and $t = 30$

c) The Area has an upper limit

the model does not.

11

a) $f(x) = \dfrac{\sin 2x}{-3+\cos 2x}$ $\qquad \dfrac{dy}{dx} = 0$ at P and Q

$y = \dfrac{\sin 2x}{-3+\cos 2x}$ use quotient rule

$\sin 2x$	$-3+\cos 2x$
$2\cos 2x$	$-2\sin 2x$

$$\frac{dy}{dx} = \frac{(-3+\cos 2x)(2\cos 2x) - (\sin 2x)(-2\sin 2x)}{(-3+\cos 2x)^2}$$

$$0 = (-3+\cos 2x)(2\cos 2x) - (\sin 2x)(-2\sin 2x)$$

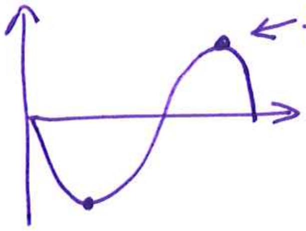
$$0 = -6\cos(2x) + 2\cos^2(2x) + 2\sin^2(2x)$$

$$0 = -6\cos(2x) + 2(\cos^2(2x) + \sin^2(2x))$$

$$0 = -6\cos(2x) + 2(1)$$

$$\cos 2x = \frac{-2}{-6} = \frac{1}{3}$$

11 b)



$$\cos 2x = \frac{1}{3}$$

1ST solution $\quad 2x = 1.23 \qquad x = 0.615$

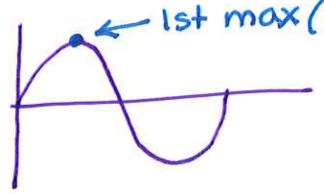
2nd solution $\quad 2x = 2\pi - 1.23 = 5.05 \qquad x = 2.525$

i) $y = f(3x) + 5$ $\qquad$ transformation $\left(\frac{x}{3}, y+5\right)$

$$\therefore \quad x = \frac{2.525}{3} = \underline{\underline{0.84}} \text{ (2dp)}$$

ii) $y = -f(x)$ $\qquad$ reflect in x axis

hence



$y = f\left(\frac{1}{4}x\right)$ $\qquad$ transformation $(4x, y)$

$$\therefore \quad x = 0.615 \times 4 = \underline{\underline{2.46}}$$

(12)

a)

2018	2019		2040
1	2	.....	23
4500			

$4500 \xrightarrow{\times 0.98}$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$a = 4500$
$r = 0.98$
$n = 23$

$$S_{23} = \frac{4500(1-0.98^{23})}{1-0.98} = 83621.86152$$

$= 83600$ tonnes

b)

1	2	3	2040 23
a	ar	ar^2	ar^{22}

$$4500(0.98)^{22} = 2885.268$$

$= 2890$ tonnes

c) Each year $800 \times (1500 \times 23) = 27.6$ million

$\left[83621.86152 - (1500 \times 23)\right] \times 600 = 29.5$ million

Total $= 57.1$ million

(13)

$x = 6\cos t$ $\qquad$ $y = 5\sin 2t$

$\frac{dx}{dt} = -6\sin t$ $\qquad$ $\frac{dy}{dt} = 10\cos 2t$

$$\text{AREA} = \int y \frac{dx}{dt}\, dt$$

$$= \int 5\sin 2t\, (-6\sin t)\, dt$$

$$= \int -30 \sin 2t \sin t\, dt$$

$$= \int -30\,(2\sin t\cos t)\sin t\, dt$$

$$= \int -60 \sin^2 t \cos t\, dt$$

$$= \int -60 \cos t\,(\sin t)^2\, dt$$

reverse chain possible answer $(\sin t)^3$

$$\begin{array}{c|c} u^3 & \sin t \\ \hline 3u^2 & \cos t \end{array}$$

$3(\sin t)^2 \cos t$

need $\times$ by $\frac{1}{3}$

$$= \left[-60\ \tfrac{1}{3}(\sin t)^3\right]$$

(13) conti

$$= \left[-20 (\sin t)^3 \right]$$

x limits	$\longrightarrow$	t limits
$x = 3$	$6\cos t$	$t = \pi/3$
$x = 0$	$6\cos t$	$t = \pi/2$

$$= \left[-20 (\sin t)^3 \right]_{\pi/2}^{\pi/3}$$

$$= -20 \left(\sin \tfrac{\pi}{3}\right)^3 - -20\left(\sin \tfrac{\pi}{2}\right)^3$$

$$= -20 \left(\frac{\sqrt{3}}{2}\right)^3 + 20 (1)^3$$

$$= -20 \left(\frac{3\sqrt{3}}{8}\right) + 20$$

$$= -\frac{15}{2}\sqrt{3} + 20$$

(14) Find limits

$$kx^2 = (kx)^{1/2}$$

square both sides

$$(kx^2)^2 = \left((kx)^{1/2}\right)^2$$

$$k^2x^4 = kx$$

$$kx^4 = x$$

$$x^4 = \frac{x}{k}$$

$$x^3 = \frac{1}{k}$$

$$x = \sqrt[3]{\frac{1}{k}}$$

$$x = \frac{1}{k^{1/3}}$$

(14) conti.

$$\int_0^{\frac{1}{k^{1/3}}} (kx)^{1/2} - \int_0^{\frac{1}{k^{1/3}}} kx^2$$

$$= \int_0^{1/k^{1/3}} k^{1/2} x^{1/2} - \int_0^{1/k^{1/3}} kx^2$$

$$= \left[\frac{k^{1/2} x^{3/2}}{3/2} \right] - \left[\frac{kx^3}{3} \right]$$

$$= \left[\frac{2 k^{1/2} x^{3/2}}{3} \right] - \left[\frac{kx^3}{3} \right]$$

sub in limits

$$= \left[\frac{2}{3} k^{1/2} \left(\frac{1}{k^{1/3}}\right)^{3/2} - 0 \right] - \left[\frac{k}{3}\left(\frac{1}{k}\right) - 0 \right]$$

$$= \left[\frac{2}{3} k^{1/2} \frac{1}{k^{3/6}} - 0 \right] - \left[\frac{1}{3} \right]$$

$$= \left[\frac{2}{3} \right] - \left[\frac{1}{3} \right]$$

$$= \frac{1}{3}$$

15

$a_2 = 2$

a)

$$a_3 = k - \frac{3k}{2} = -\frac{1}{2}k$$

$$a_4 = k - \frac{3k}{-\frac{1}{2}k} = k + 6$$

$$a_5 = k - \frac{3k}{k+6}$$

periodic order of 3 (repeats every 3 terms)

a_2	a_3	a_4	a_5	a_6	a_7	a_8	
2			2			2	etc....

$$2 = k - \frac{3k}{k+6}$$

$$2 = \frac{k}{k+6} - \frac{3k}{k+6}$$

$$2 = \frac{k(k+6) - 3k}{k+6}$$

$$2k + 12 = k^2 + 6k - 3k$$

$$0 = k^2 + k - 12$$

15

b) $\sum_{r=1}^{121} a_r$

$k^2+k-12=0$

$(k+4)(k-3)=0$

$k=-4$ or 3

$k=3$

a_2	a_3	a_4	a_5	a_6	a_7
	$3-\frac{9}{2}$	$3\div\frac{9}{}$			
2	-1.5	9	2	-1.5	9

hence $a_1 = 9$

$k=-4$

a_2	a_3	a_4	a_5	a_6	a_7
	$-4+\frac{12}{2}$	$-4+\frac{12}{2}$			
2	2	2	2	2	2

hence $a_1 = 2$ Reject $a_1 = a_2$

$121 \div 3 = 40\frac{1}{3}$

$(2+-1.5+9)\times 40 + 9 = 389$