AS and A level Further Mathematics Practice Paper – Proof – Mark scheme

Question	Scheme	Marks
1(a)	If $n = 1$, $\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^1 - 1) & 5^1 \end{pmatrix}$ so **true** for $\boldsymbol{n = 1}$	B1
	Assume result true for $n = k$	
	$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k - 1) & 5^k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k - 1) & 5^k \end{pmatrix}$	M1
	$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k - 1) - 5^k & 5 \times 5^k \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ -1 - 5.\frac{1}{4}(5^k - 1) & 5 \times 5^k \end{pmatrix}$	M1 A1
	$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}5^k + \frac{1}{4} - 5^k & 5^{k+1} \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ -1 - \frac{1}{4}5^{k+1} + \frac{5}{4} & 5^{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^{k+1} - 1) & 5^{k+1} \end{pmatrix}$	A1
	True for $\boldsymbol{n = k + 1}$ if **true** for $\boldsymbol{n = k}$, (and **true** for $\boldsymbol{n = 1}$) so **true** by induction for all $\boldsymbol{n \in Z^+}$	A1cso
		(6)
1(b)	If $n = 1$, $\sum_{r=1}^{n}(2r-1)^2 = 1$ and $\frac{1}{3}n(4n^2 - 1) = 1$, so **true** for $\boldsymbol{n = 1}$.	B1
	Assume result true for $n = k$ so $\sum_{r=1}^{k+1}(2r-1)^2 = \frac{1}{3}k(4k^2 - 1) + (2(k+1) - 1)^2$	M1
	$= \sum_{r=1}^{k+1}(2r-1)^2 = \frac{1}{3}(2k+1)\{(2k^2 - k) + (3(2k+1))\}$	M1 A1
	$= \frac{1}{3}(2k+1)\{(2k^2 + 5k + 3)\}$ or $\frac{1}{3}(k+1)(4k^2 + 8k + 3)$ or $\frac{1}{3}((2k+3)(2k^2 + 3k + 1)\}$	
	$= \frac{1}{3}(k+1)(2k+1)(2k+3) \quad = \frac{1}{3}(k+1)(4(k+1)^2 - 1)$	dA1
	True for $\boldsymbol{n = k + 1}$ if **true** for $\boldsymbol{n = k}$, (and **true** for $\boldsymbol{n = 1}$) so **true** by induction for all $\boldsymbol{n \in Z^+}$	A1cso
		(6)
		(12 marks)

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Question	Scheme	Marks
2(i)	$u_{n+2} = 6u_{n+1} - 9u_n$, n Ö1, $u_1 = 6$, $u_2 = 27$; $u_n = 3^n(n+1)$ $n = 1$; $u_1 = 3(2) = 6$ $n = 2$; $u_2 = 3^2(2+1) = 27$ So u_n is true when $n = 1$ and $n = 2$.	 B1
	Assume that $u_k = 3^k(k+1)$ and $u_{k+1} = 3^{k+1}(k+2)$ are true. Then $u_{k+2} = 6u_{k+1} - 9u_k$ $= 6(3^{k+1})(k+2) - 9(3^k)(k+1)$ $= 2(3^{k+2})(k+2) - (3^{k+2})(k+1)$ $= (3^{k+2})(2k+4-k-1)$ $= (3^{k+2})(k+3)$ $= (3^{k+2})(k+2+1)$	 M1A1 M1 A1
	If the result is **true for *n* = *k* and *n* = *k*+1** then it is now **true for *n* = *k* + 2**. As it is **true for *n* = 1 and *n* = 2** then it is **true for all *n*** $\in \mathbb{Z}^+$	A1 cso **(6)**
2(ii)	$\mathrm{f}(n) = 3^{3n-2} + 2^{3n+1}$ is divisible by 19 $\mathrm{f}(1) = 3^1 + 2^4 = 19$ {which is divisible by 19}. { $\therefore \mathrm{f}(n)$ is divisible by 19 when $n = 1$ }	 B1
	{Assume that for $n = k$, $\mathrm{f}(k) = 3^{3k-2} + 2^{3k+1}$ is divisible by 19 for $k \in \mathbb{Z}^+$} $\mathrm{f}(k+1) - \mathrm{f}(k) = 3^{3(k+1)-2} + 2^{3(k+1)+1} - (3^{3k-2} + 2^{3k+1})$ $\mathrm{f}(k+1) - \mathrm{f}(k) = 27(3^{3k-2}) + 8(2^{3k+1}) - (3^{3k-2} + 2^{3k+1})$ $\mathrm{f}(k+1) - \mathrm{f}(k) = 26(3^{3k-2}) + 7(2^{3k+1})$ $= 7(3^{3k-2} + 2^{3k+1}) + 19(3^{3k-2})$ **or** $= 26(3^{3k-2} + 2^{3k+1}) - 19(2^{3k+1})$ $= 7\mathrm{f}(k) + 19(3^{3k-2})$ **or** $= 26\mathrm{f}(k) - 19(2^{3k+1})$ $\therefore \mathrm{f}(k+1) = 8\mathrm{f}(k) + 19(3^{3k-2})$ **or** $\mathrm{f}(k+1) = 27\mathrm{f}(k) - 19(2^{3k+1})$ { $\therefore \mathrm{f}(k+1) = 8\mathrm{f}(k) + 19(3^{3k-2})$ is divisible by 19 as both $8\mathrm{f}(k)$ and $19(3^{3k-2})$ are both divisible by 19}	 M1 A1 A1 dM1
	If the result is **true for *n* = *k***, then it is now **true for *n* = *k* + 1**. As the result has shown to be **true for *n* = 1**,then the result is **true for all *n*** $\in \mathbb{Z}^+$	A1 cso **(6)**
		(12 marks)

Question	Scheme	Marks
3	$\mathrm{f}(n) = 8^n - 2^n$ is divisible by 6.	
	$\mathrm{f}(1) = 8^1 - 2^1 = 6$,	B1
	Assume that for $n = k$, $\mathrm{f}(k) = 8^k - 2^k$ is divisible by 6.	
	$\mathrm{f}(k+1) - \mathrm{f}(k) = 8^{k+1} - 2^{k+1} - \left(8^k - 2^k\right)$	M1
	$= 8^k\left(8-1\right) + 2^k\left(1-2\right) = 7\times 8^k - 2^k$	
	$= 6\times 8^k + 8^k - 2^k \left(= 6\times 8^k + \mathrm{f}\left(k\right)\right)$	M1A1
	$\mathrm{f}(k+1) = 6\times 8^k + 2\mathrm{f}\left(k\right)$	A1
	If the result is **true for** $n = k$, then it is now **true for** $n = k+1$. As the result has been shown to be **true for** $n = 1$, then the result is **true for all** $\boldsymbol{n} \in \mathbb{Z}^+$	A1cso
		(6marks)

Question	Scheme		Marks
4(a)	$n=1;\quad \text{LHS} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^1 = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ $\text{RHS} = \begin{pmatrix} 3^1 & 0 \\ 3(3^1 - 1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ As $\text{LHS} = \text{RHS}$, the matrix result is true for $n = 1$.	Check to see that the result is true for $n = 1$.	B1
	Assume that the matrix equation is true for $n = k$, ie. $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}$		
	With $n = k+1$ the matrix equation becomes $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ $= \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ or $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$	$\begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}$ by $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$	M1
	$= \begin{pmatrix} 3^{k+1} + 0 & 0 + 0 \\ 9(3^k - 1) + 6 & 0 + 1 \end{pmatrix}$ or $= \begin{pmatrix} 3^{k+1} + 0 & 0 + 0 \\ 6.3^k + 3(3^k - 1) & 0 + 1 \end{pmatrix}$	Correct unsimplified matrix with no errors seen.	A1
	$= \begin{pmatrix} 3^{k+1} & 0 \\ 9(3^k) - 3 & 1 \end{pmatrix} \qquad = \begin{pmatrix} 3^{k+1} & 0 \\ 3\left(3(3^k) - 1\right) & 1 \end{pmatrix}$		
	$= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1} - 1) & 1 \end{pmatrix}$	Manipulates so that $k \to k+1$ on at least one term.	dM1
		Correct result with no errors seen with some working between this and the previous A1	A1
	If the result is true for $n = k$(1) then it is now true for $n = k+1$. (2) As the result has shown to be true for $n = 1$,(3)then the result is true for all *n*. (4)	Correct conclusion with all previous marks earned	A1 cso
			(6)

Question	Scheme		Marks
4(b)	$\mathrm{f}(1) = 7^{2-1} + 5 = 7 + 5 = 12$, {which is divisible by 12}. $\{\therefore \mathrm{f}(n)$ is divisible by 12 when $n = 1.\}$	Shows that $\mathrm{f}(1) = 12$.	B1
	Assume that for $n = k$, $\mathrm{f}(k) = 7^{2k-1} + 5$ is divisible by 12 for $k \in \mathbb{Z}^+$.		
	So, $\mathrm{f}(k+1) = 7^{2(k+1)-1} + 5$	Correct unsimplified expression for $\mathrm{f}(k+1)$.	B1
	giving, $\mathrm{f}(k+1) = 7^{2k+1} + 5$		
	$\therefore \mathrm{f}(k+1) - \mathrm{f}(k) = \left(7^{2k+1} + 5\right) - \left(7^{2k-1} + 5\right)$	Applies $\mathrm{f}(k+1) - \mathrm{f}(k)$. No simplification is necessary and condone missing brackets.	M1
	$= 7^{2k+1} - 7^{2k-1}$		
	$= 7^{2k-1}\left(7^2 - 1\right)$	Attempting to isolate 7^{2k-1}	M1
	$= 48\left(7^{2k-1}\right)$	$48\left(7^{2k-1}\right)$	A1cso
	$\therefore \mathrm{f}(k+1) = \mathrm{f}(k) + 48\left(7^{2k-1}\right)$, which is divisible by 12 as both $\mathrm{f}(k)$ and $48\left(7^{2k-1}\right)$ are both divisible by 12.(1) If the result is true for $n = k$, (2) then it is now true for $n = k+1$. (3) As the result has shown to be true for $n = 1$,(4)then the result is true for all n. (5).	Correct conclusion with no incorrect work. Don't condone missing brackets.	A1 cso
			(6)
			(12 marks)

Question	**Scheme**		**Marks**
5	$u_{n+1} = 4u_n + 2$, $u_1 = 2$ and $u_n = \frac{2}{3}(4^n - 1)$		
	$n = 1$; $u_1 = \frac{2}{3}(4^1 - 1) = \frac{2}{3}(3) = 2$ So u_n is true when $n = 1$.	Check that $u_n = \frac{2}{3}(4^n - 1)$ yields 2 when $n = 1$.	B1
	Assume that for $n = k$ that, $u_k = \frac{2}{3}(4^k - 1)$ is true for $k \in \mathbb{Z}^+$.		
	Then $u_{k+1} = 4u_k + 2$		
	$= 4\left(\frac{2}{3}(4^k - 1)\right) + 2$	Substituting $u_k = \frac{2}{3}(4^k - 1)$ into $u_{n+1} = 4u_n + 2$.	M1
	$= \frac{8}{3}(4)^k - \frac{8}{3} + 2$	An attempt to multiply out the brackets by 4 or $\frac{8}{3}$	M1
	$= \frac{2}{3}(4)(4)^k - \frac{2}{3}$		
	$= \frac{2}{3}4^{k+1} - \frac{2}{3}$		
	$= \frac{2}{3}(4^{k+1} - 1)$	$\frac{2}{3}(4^{k+1} - 1)$	A1
	Therefore, the general statement, $u_n = \frac{2}{3}(4^n - 1)$ is true when $n = k + 1$. (As u_n is true for $n = 1$,) then u_n is true for all positive integers by mathematical induction	Require 'True when n=1', 'Assume true when *n*=*k*' and 'True when $n = k + 1$' then true for all *n* o.e.	A1
			(5 marks)

Question	Scheme	Marks
6	$\mathrm{f}(n) = 2^{2n-1} + 3^{2n-1}$ is divisible by 5.	
	$\mathrm{f}(1) = 2^1 + 3^1 = 5$,	B1
	Assume that for $n = k$,	
	$\mathrm{f}(k) = 2^{2k-1} + 3^{2k-1}$ is divisible by 5 for $k \in \mathbb{Z}^+$.	
	$\mathrm{f}(k+1) - \mathrm{f}(k) = 2^{2(k+1)-1} + 3^{2(k+1)-1} - \left(2^{2k-1} + 3^{2k-1}\right)$	M1A1
	$= 2^{2k+1} + 3^{2k+1} - 2^{2k-1} - 3^{2k-1}$	
	$= 2^{2k-1+2} + 3^{2k-1+2} - 2^{2k-1} - 3^{2k-1}$	
	$= 4\left(2^{2k-1}\right) + 9\left(3^{2k-1}\right) - 2^{2k-1} - 3^{2k-1}$	M1
	$= 3\left(2^{2k-1}\right) + 8\left(3^{2k-1}\right)$	
	$= 3\left(2^{2k-1}\right) + 3\left(3^{2k-1}\right) + 5\left(3^{2k-1}\right)$	
	$= 3\mathrm{f}(k) + 5\left(3^{2k-1}\right)$	
	$\therefore \mathrm{f}(k+1) = 4\mathrm{f}(k) + 5\left(3^{2k-1}\right)$ or $4(2^{2k-1} + 3^{2k-1}) + 5\left(3^{2k-1}\right)$	A1
	If the result is true for $n = k$, then it is now true for $n = k+1$. As the result has shown to be true for $n = 1$, then the result is true for all n.	A1 cso
		(6 marks)

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	Source paper	Question number	New spec references	Question description	New AOs
1	FP1 2015	6		Proof - induction, Matrices	1.1b, 2.1
2	FP1 2017	9		Proof	1.1b, 2.1
3	FP1 2014	9		Proof	1.1b, 2.1
4	FP1 2011	9		Proof	1.1b, 2.1
5	FP1 2011	9		Proof	1.1b, 2.1
6	FP1 2012	10		Proof	1.1b, 2.1