PURE 2

MOCK PAPER 1

JANUARY 2019

WORKED SOLUTIONS

1

a) $\text{Area} = \frac{1}{2}(0.5)\left[1 + e^{0.8} + 2\left(e^{0.05} + e^{0.2} + e^{0.45}\right)\right]$

$= 2.73 \text{ (2dp)} \quad \therefore \int e^{\frac{1}{5}x^2} \approx 2.73$

b) i) $\int_0^2 4 + e^{\frac{1}{5}x^2} \, dx$

$= \int_0^2 4 + \int_0^2 e^{1/5 x^2}$

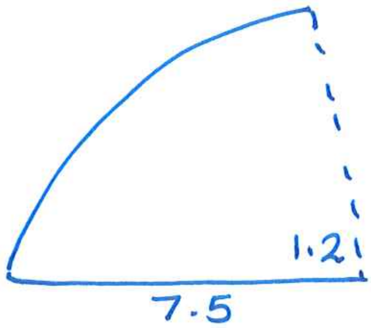
$= \left[4x\right]_0^2 + 2.73$

$= 8 + 2.73 = 10.73$

ii) $\int_1^3 e^{\frac{1}{5}(x-1)^2} = \int_0^2 e^{\frac{1}{5}(x)^2}$

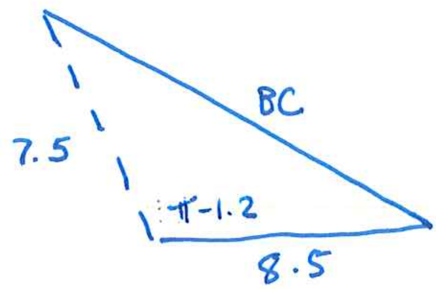
$= 2.73$

2



Arc length $= r\theta$

$= 7.5(1.2) = 9$



$BC^2 = 7.5^2 + 8.5^2 - 2(7.5)(8.5)\cos(\pi - 1.2)$

$BC = 13.21743597$

Total perimeter $= 9 + 7.5 + 8.5 + 13.21743$

$= 38.2$ (1dp)

(3)

$$3x^2 + k = 5x + 2$$

$$3x^2 - 5x + k - 2 = 0$$

$$a = 3$$
$$b = -5$$
$$c = k - 2$$

no real roots $\therefore$ $b^2 - 4ac < 0$

$$(-5)^2 - 4(3)(k-2) < 0$$

$$25 - 12(k-2) < 0$$

$$25 - 12k + 24 < 0$$

$$49 < 12k$$

$$k > \frac{49}{12}$$

4

a) If x is very big

$$f(x) = \frac{12x}{3x+4} \approx \frac{12x}{3x} = 4$$

If x is the smallest value $(x = 0)$

$$f(x) = \frac{0}{4} = 0$$

$$0 \leq f(x) < 4$$

b) $y = \dfrac{12x}{3x+4}$

$$y3x + 4y = 12x$$

$$y3x - 12x = -4y$$

$$x(3y - 12) = -4y$$

$$x = \frac{-4y}{3y-12}$$

$$y = \frac{-4x}{3x-12} \quad \text{or} \quad \frac{4x}{12-3x}$$

4c

$$ff(x) = \frac{12\left(\frac{12x}{3x+4}\right)}{3\left(\frac{12x}{3x+4}\right)+4}$$

$$= \frac{\frac{144x}{3x+4}}{\frac{36x}{3x+4} + 4}$$

$$= \frac{\frac{144x}{3x+4}}{\frac{36x}{3x+4} + \frac{4(3x+4)}{3x+4}}$$

$$= \frac{\frac{144x}{3x+4}}{\frac{48x+16}{3x+4}}$$

$$= \frac{144x}{48x+16}$$

$$= \frac{9x}{3x+1}$$

4

d) $$\frac{9x}{3x+1} = \frac{7}{2}$$

$$18x = 21x + 7$$

$$-3x = 7$$

$$x = \frac{7}{-3}$$

$$x \geqslant 0$$

$\therefore$ no solutions

(5) $f(x) = 4x^2 - 5x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{4(x+h)^2 - 5(x+h) - [4(2)^2 - 5(2)]}{h}$$

At point A $x = 2$
$(2, 6)$

$$f'(x) = \lim_{h \to 0} \frac{4(2+h)^2 - 5(2+h) - [6]}{h}$$

$$= \lim_{h \to 0} \frac{4h^2 + 16h + 4 - 10 - 5h - 6}{h}$$

$$= \lim_{h \to 0} \frac{4h^2 + 11h}{h} = 4h + 11$$

$$f'(x) = 4h + 11$$

$$f'(x) = 11$$

$h \to 0$
(ie $h = 0$)

(6)

a) x limits $\displaystyle\int_0^2 \frac{6}{e^{1/2x}+4}\,dx$

change limits
$u = e^{\frac{1}{2}x}$

u limits $\displaystyle\int_1^e \frac{6}{u+4}\,dx$

$$u = e^{\frac{1}{2}x}$$

$$\frac{du}{dx} = \frac{1}{2}e^{\frac{1}{2}x}$$

$$du = \frac{e^{1/2x}}{2}\,dx$$

$$\frac{2}{e^{1/2x}}\,du = dx$$

$$\int \frac{6}{u+4} \cdot \frac{2}{e^{1/2x}}\,du$$

$$\int \frac{12}{(u+4)u}\,du$$

(6)

b) $$\int \frac{12}{u(u+4)} = \frac{A}{u} + \frac{B}{u+4}$$

$$\therefore \quad 12 = A(u+4) + Bu$$

let $u = -4$ $\qquad 12 = -4B$

$\qquad B = -3$

let $u = 0$ $\qquad 12 = 4A$

$\qquad A = 3$

$$\int \frac{3}{u} - \frac{3}{u+4} \, du$$

$$= \left[3\ln u - 3\ln(u+4) \right]$$

x limits $\qquad$ u limits

$$2 \xrightarrow{u=e^{\frac{1}{2}x}} e$$

$$0 \longrightarrow 1$$

⑥

b) conti

$$\left[3\ln u - 3\ln(u+4)\right]_1^e$$

$$= \left[3\ln e - 3\ln(e+4)\right] - \left[3\ln 1 - 3\ln 5\right]$$

$$= \left[3\ln e - 3\ln(e+4)\right] - \left[0 - 3\ln 5\right]$$

$$= \left[3\ln e - 3\ln(e+4)\right] + 3\ln 5$$

$$= \ln e^3 + \ln 5^3 - \ln(e+4)^3$$

$$= \ln \frac{e^3 5^3}{(e+4)^3}$$

$$= \ln\left(\frac{5e}{e+4}\right)^3$$

⑦

a) $3\sin\theta - 4\cos\theta$

compare to addition rule

$\sin\theta\cos\theta - \cos\theta\sin\theta$

hence

$\cos\theta = 3$ $\sin\theta = 4$

$\tan\theta = \frac{4}{3}$ $\theta = 53.13°$

$R = \sqrt{3^2+4^2} = \sqrt{25} = 5$

b) i $G = 17 + 3\sin(15t) - 4\cos(15t)$

$G = 17 + \underbrace{5\sin(15t - 53.13)}_{\text{max } 5}$

$G = 17 + 5 = 22$

7

b ii) $20 = 17 + 5\sin(15t - 53.13)$

$\sin(15t - 53.13) = \frac{3}{5}$

for 5am $t = 0$

for midday $t = 7$ $\therefore$ t must be > 7

1ST SOLUTION $15t - 53.13 = \underbrace{36.87}_{\sin^{-1}(\frac{3}{5})}$

$t = 6$ Not > 7

2ND SOLUTION $15t - 53.13 = \underbrace{180 - 36.87}_{180 - \text{1ST SOLUTION}}$

$t = 13.084$

hence $t = 13$ hours 5 mins (nearest min)

Time 6.05 pm

(8)

i) $y^2 - 4y + 7$

complete the square

$(y-2)^2 - 4 + 7$

$(y-2)^2 + 3$

$(y-2)^2 \geq 0$

$\therefore$ $y^2 - 4y + 7$ is positive for all real values of y

ii)

$x = -1$ $\quad e^{-3} < e^{-2}$ not true Bobby's claim

$x = 2$ $\quad e^{6} > e^{4}$ Bobby's claim true

Therefore Bobby's claim is sometimes true

8

iii) $n = \text{odd} = 2k+1$

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$
$$= \text{Even} \quad +1$$
$$= \text{odd}$$

$n = \text{even} = 2k$

$$n^2 = (2k)^2 = 4k^2$$
$$= 2(2k^2)$$
$$= \text{Even}$$

Elsa claim is true

iv) $\underset{\text{irrational}}{(\pi)} + \underset{\text{irrational}}{(9-\pi)} = \underset{\text{rational}}{9}$ claim fails

$\underset{\text{irrational}}{(\pi)} + \underset{\text{irrational}}{(9+\pi)} = \underset{\text{irrational}}{2\pi+9}$ clain true

Yings claim is sometimes true

(9)

a) $$\frac{\sin x}{1-\cos x} + \frac{1-\cos x}{\sin x}$$

$$= \frac{(\sin x)^2}{(1-\cos x)(\sin x)} + \frac{(1-\cos x)^2}{(1-\cos x)(\sin x)}$$

$$= \frac{(\sin x)^2 + (1-\cos x)^2}{(1-\cos x)(\sin x)}$$

$$= \frac{\sin^2 x + 1 - 2\cos x + \cos^2 x}{(1-\cos x)\sin x}$$

$\sin^2 x + \cos^2 x = 1$

$$= \frac{2 - 2\cos x}{(1-\cos x)\sin x}$$

$$= \frac{2(1-\cos x)}{(1-\cos x)\sin x}$$

$$= \frac{2}{\sin x}$$

$$= 2\operatorname{cosec} x$$

$\therefore\ k = 2$

9

b) $2\operatorname{cosec} x = 1.6$

$$\frac{2}{\sin x} = 1.6$$

$$\sin x = \frac{2}{1.6}$$

$$\sin x = 1.25$$

$-1 \leq \sin x \leq 1$

$\therefore$ no real solutions

(10) volume ÷ rate of flow $\quad h = 24$

a) $\dfrac{4\pi h(h+6)}{80\pi}$

$= \dfrac{4\pi(24)(24+6)}{80\pi}$

$= \dfrac{2880\pi}{80\pi}$

$= 36$ sec

b) Find $\dfrac{dh}{dt}$ $\quad \dfrac{dV}{dt} = 80\pi$

$80\pi \times 8$

when $t = 8 \quad V = 640\pi$

$640\pi = 4\pi h(h+6)$

$h^2 + 6h - 160 = 0$

$h = 10$ or -16

$\therefore h = 10$

$\dfrac{dh}{dt} = \dfrac{dV}{dt} \times \dfrac{dh}{dV}$

(10)

b) conti

$$\frac{dh}{dt} = \frac{dv}{dt} \times \frac{dh}{dv}$$

$$v = 4\pi h^2 + 24\pi h$$

$$\frac{dv}{dh} = 8\pi h + 24\pi$$

$$\frac{dv}{dt} = 80\pi$$

$$\frac{dh}{dt} = 80\pi \times \frac{1}{8\pi h + 24\pi}$$

$h = 10$

$$\frac{dh}{dt} = 80\pi \times \frac{1}{80\pi + 24\pi}$$

$$= \frac{80\pi}{104\pi}$$

$$= \frac{10}{13}$$

(11)

i) $y = a^x$

$\ln y = \ln a^x$

$\ln y = x \ln a$

implict

$\frac{1}{y}\frac{dy}{dx} = \ln a$

$\frac{dy}{dx} = y \ln a$

we know $y = a^x$

$\frac{dy}{dx} = a^x \ln a$

11 ii) $x = 2\tan y$

implict

$$1 = 2\sec^2 y \frac{dy}{dx}$$

$$1 = 2(1+\tan^2 y)\frac{dy}{dx}$$

$$x = 2\tan y$$

$$\frac{x}{2} = \tan y$$

$$\frac{x^2}{4} = \tan^2 y$$

$$1 = 2\left(1+\frac{x^2}{4}\right)\frac{dy}{dx}$$

$$1 = \left(2+\frac{x^2}{2}\right)\frac{dy}{dx}$$

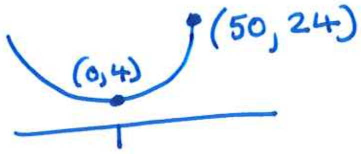
$$1 = \left(\frac{4+x^2}{2}\right)\frac{dy}{dx}$$

$$\frac{2}{4+x^2} = \frac{dy}{dx}$$

(12)

a) $y = ax^2 + c$

$y = ax^2 + 4$



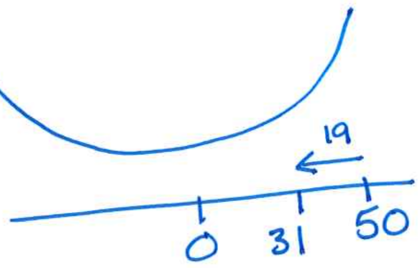
$= a(50)^2 + 4$

$24 = 2500a + 4$

$a = \frac{1}{125}$

$y = \frac{1}{125}x^2 + 4$

b)



defect $x = 31$

$y = \frac{1}{125}(31)^2 + 4$

$y = 11.688$

less than 12 so can safely inspect

c) thickness of cable not considerd / or weather may effect shape

13

a) $$\sum_{n=1}^{11} \ln(p^n)$$

$$= \ln(p^1) + \ln(p^2) + \ln(p^3) + \ldots + \ln(p^{11})$$

$$= \ln p^{66}$$

$$= 66 \ln p$$

b) $$\sum_{n=1}^{11} \ln(8p^n)$$

$$= \ln 8p + \ln 8p^2 + \ln 8p^3 + \ldots + \ln 8p^{11}$$

$$= \ln 8 + \ln p + \ln 8 + \ln p^2 + \ldots \text{etc}$$

$$= 11 \ln 8 + \ln p^{66}$$

$$= 11 \ln 2^3 + \ln p^{66}$$

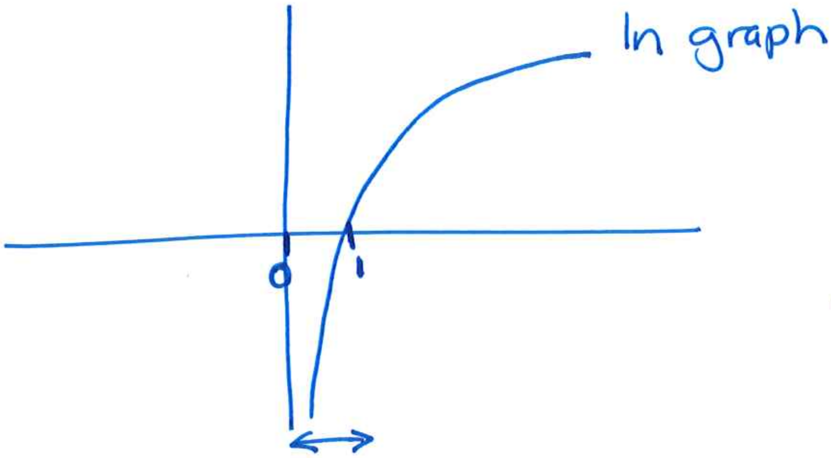
$$= 33 \ln 2 + \ln p^{66}$$

$$= \ln 2^{33} + \ln p^{66} = \ln 2^{33} p^{66} = \ln (2p^2)^{33}$$

(13)

c) $33 \ln(2p^2) < 0$

$\ln(2p^2) < 0$



$0 < 2p^2 < 1$

$0 < p^2 < \frac{1}{2}$

$0 < p < \frac{1}{\sqrt{2}}$

set notation

$\left\{ p : 0 < p < \frac{1}{\sqrt{2}} \right\}$

(14)

$$y = 4xe^{-2x}$$

$4x$	e^{-2x}
4	$-2e^{-2x}$

$$\frac{dy}{dx} = 4e^{-2x} - 8xe^{-2x}$$

AT $P(1, 4e^{-2})$

$$\frac{dy}{dx} = 4e^{-2} - 8e^{-2}$$

$$\frac{d}{dx} = -4e^{-2}$$

tangent gradient $\frac{-4}{e^2}$

normal gradient $\frac{e^2}{4}$

normal equation $P(1, 4e^{-2})$ $y - y_1 = m(x - x_1)$

$$y - 4e^{-2} = \frac{e^2}{4}(x-1)$$

when $y=0$

$$x = 1 - 16e^{-4}$$

(14) conti

Area under curve

$$\int_0^1 4xe^{-2x}$$

$4x$	$-\frac{1}{2}e^{-2x}$
4	e^{-2x}

$$\int_0^1 4xe^{-2x} = -2xe^{-2x} - \int -2e^{-2x}\,dx$$

$$= \left[-2xe^{-2x} - e^{-2x}\right]_0^1$$

$$= \left[-2e^{-2} - e^{-2}\right] - \left[0 - 1\right]$$

$$= \left[-3e^{-2}\right] + \left[1\right]$$

$$= 1 - 3e^{-2}$$

Area Triangle

$$\frac{1}{2}(16e^{-4})(4e^{-2}) = 32e^{-6}$$

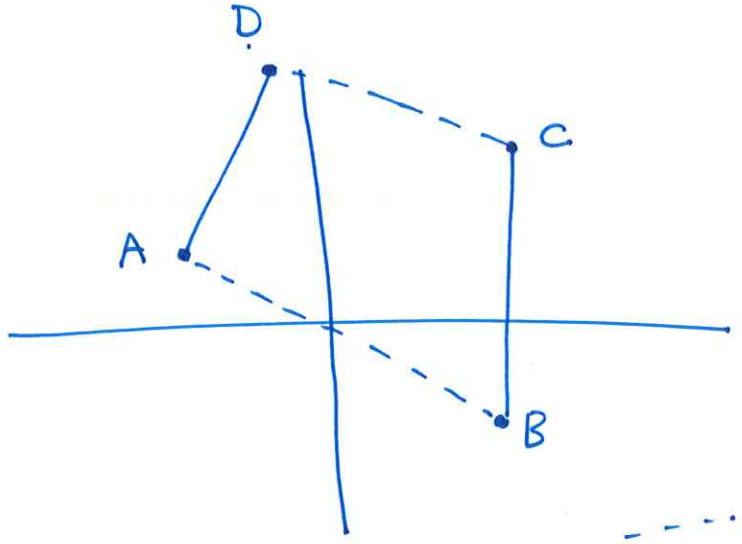
$$R \text{ Area} = 1 - 3e^{-2} - 32e^{-6}$$

15

a) $OD = OA + AD$

$$= \begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 3 \end{pmatrix}$$

b)



$$OC = OB + BC$$
$$= \begin{pmatrix} 3 \\ -1 \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ 6 \\ -7 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 5 \\ p-7 \end{pmatrix}$$

$$DC = DO + OC$$

$$= \begin{pmatrix} 1 \\ -7 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ p-7 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -2 \\ p-10 \end{pmatrix}$$

$$AB = AO + OB$$

$$= \begin{pmatrix} 3 \\ -2 \\ -7 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ p \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -3 \\ p-7 \end{pmatrix}$$

15 conti

$$DC = \begin{pmatrix} 4 \\ -2 \\ p-10 \end{pmatrix} \qquad AB = \begin{pmatrix} 6 \\ -3 \\ p-7 \end{pmatrix}$$

same gradient

$$\frac{-2}{p-10} = \frac{-3}{p-7}$$

$$-2p+14 = -3p+30$$

$$p = 16$$