Year 2 Pure Chapter 11 – Integration

Standard Functions (Must Learn)

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c$$

$$\int (ax+b)^n \, dx = \frac{1}{a(n+1)}(ax+b)^{n+1}$$

$$\int \frac{1}{ax} \, dx = \frac{1}{a}\ln|ax| + c$$

---

denominator to power of 1

$$\int \frac{1}{(ax+b)} \, dx = \frac{1}{a}\ln|ax+b| + c$$

---

denominator to power of that is not 1

$$\int \frac{1}{(ax+b)^n} \, dx = \frac{1}{a(-n+1)}(ax+b)^{-n+1}$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + c$$

$$\int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + c$$

$$\int \sin(ax+b) \, dx = -\frac{1}{a}\cos(ax+b) + c$$

Trig Function (Formula Booklet)

Integration

$$\int \sec^2 kx \, dx = \frac{1}{k}\tan kx + c$$

$$\int \tan kx \, dx = \frac{1}{k}\ln[\sec kx] + c$$

$$\int \cot kx \, dx = \frac{1}{k}\ln|\sin kx| + c$$

$$\int \operatorname{cosec} kx \, dx$$

$$= -\frac{1}{k}\ln|\operatorname{cosec} kx + \cot kx| + c$$

$$= \frac{1}{k}\ln\left|\tan\left(\frac{1}{2}kx\right)\right| + c$$

$$\int \sec kx \, dx$$

$$= \frac{1}{k}\ln|\sec kx + \tan kx| + c$$

$$= \frac{1}{k}\ln\left|\tan\left(\frac{1}{2}kx + \frac{1}{4}\pi\right)\right| + c$$

Differentiation

First Principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\mathbf{f(x)} \xleftarrow{\text{Integration}} \mathbf{f'(x)}$$

$f(x)$	$f'(x)$
$\frac{1}{k}\tan kx + C$	$k\sec^2 kx$
$\frac{1}{k}\sec kx + C$	$k\sec kx \tan kx$
$\frac{1}{k}\cot kx + C$	$-k\operatorname{cosec}^2 kx$
$\frac{1}{k}\operatorname{cosec} kx + C$	$-k\operatorname{cosec} kx \cot kx$

Reverse Chain Rule
(Multiplying Functions)

$$\int f'(x)\big(f(x)\big)^n$$

Possible answer

$$[f(x)]^{n+1}$$

Differentiate possible answer
to see if you need a multiplier

Reverse Chain Rule
(Dividing Functions)

$$\int \frac{f'(x)}{f(x)}\,dx$$

Possible answer

$$\ln|f(x)| + c$$

Differentiate possible answer
to see if you need a multiplier

By Parts
(Multiplying Functions)

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Given in the formula booklet.

u will always be the $\boldsymbol{x^n}$ term
UNLESS one term is $\boldsymbol{\ln x}$.

Special Case:

$$\int \ln x \;\, dx = \int (1)(\ln x)\;\, dx$$

Now do by parts

By Substitution

Rewrite the question and the limits in terms of u

$$\int_1^2 x\sqrt{2x+5}\,dx$$

Let $u = 2x + 5$

Rearrange $x = \frac{u-5}{2}$

Differentiate $\frac{du}{dx} = 2$

Rearrange $dx = \frac{1}{2}du$

Change Limits using,
$u = 2x + 5$

When $x = 2$ then $u = 9$
When $x = 1$ then $u = 7$

Now substitute,

$$\int_7^9 \left(\frac{u-5}{2}\right)\sqrt{u}\,\frac{1}{2}du$$

Tidy up,

$$\int_7^9 \frac{u^{\frac{3}{2}}}{4} - \frac{5u^{\frac{1}{2}}}{4}du$$

Now integrate,
and substitute in limits.

Year 2 Pure Chapter 11 – Integration

Partial Fractions

Take a single fraction and split it into two or more fractions then integrate each individual fraction.

$$\int \frac{A}{3x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}\ dx$$

Numerator: Power of 1

$$\int \frac{A}{3x+1} = \frac{1}{3} A \ln[x-2]$$

$$\int \frac{B}{x-2} = B \ln[x-2]$$

Numerator: Power not 1

$$\int \frac{C}{(x-2)^2} = \frac{C(x-2)^{-1}}{-1}$$

$$= \frac{1}{3} A \ln[x-2] + \frac{1}{3} A \ln[x-2] + \frac{C(x-2)^{-1}}{-1} + c$$

Before doing partial fractions always check to see if you have a **top-heavy fraction**. Is the highest power of the numerator equal or greater than the highest power of the denominator?

$$\int \frac{9x^2-3x+2}{9x^2-4}$$

If yes then do algebraic long division,

$$= \int 1 + \frac{6-3x}{9x^2-4}$$

Now partial fractions,

$$= \int 1 + \frac{1}{3x-2} - \frac{2}{3x-2}$$

Now integrate.

Differential Equations

(Mix of variable and derivatives)

$$\frac{dy}{dx} = xy$$

Split

$$\int \frac{1}{y}\ dy = \int x\ dx$$

Integrate

$$\ln[y] = \frac{x^2}{2} + c$$

Make y the subject

$$y = e^{\frac{x^2}{2}+c}$$

Simplify if possible

$$y = e^c e^{\frac{x^2}{2}+c}$$

$$y = Ae^{\frac{x^2}{2}+c}$$

Parametric Equations

Area Under the Graph

Cartesian Equation

$$Area = \int y\ dx$$

Parametric Equation

$$Area = \int y \frac{dx}{dt}\ dt$$

Don't forget to change the limit

x *limits* to t *limits*

Trapezium Rule

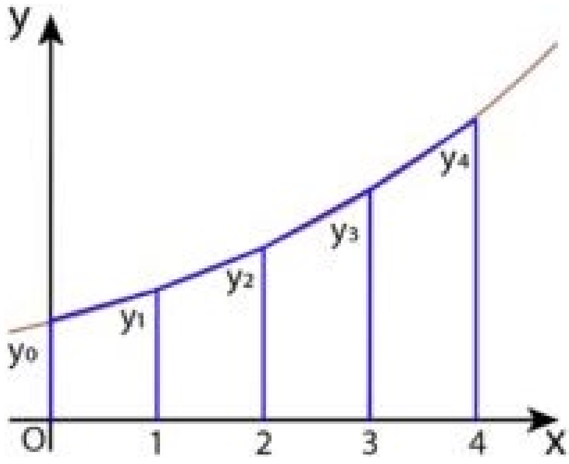
(Estimate Area Under the Graph)

$$\frac{1}{2}h[(y_o + y_1) + 2(y_1 + y_1 + \cdots \ldots + y_{n-1})]$$

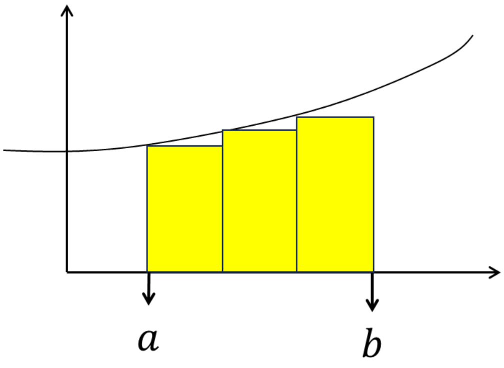
h = width of the trapeziums

$y_o + y_1$ = First and last height

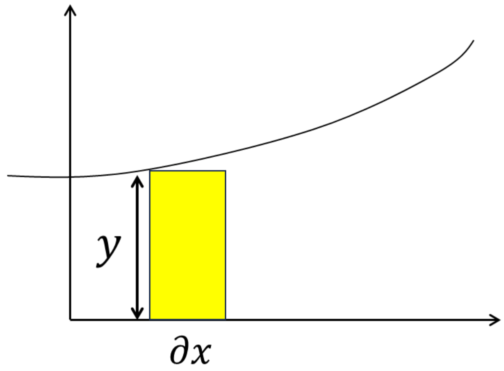
$y_1 + y_1 + \cdots \ldots + y_{n-1}$ = All other heights



Integration as a Limit of a sum



$$\lim_{\partial x \to 0} \sum_{x=a}^{b} Area\ of\ Rectangle$$



$$\lim_{\partial x \to 0} \sum_{x=a}^{b} y\ \partial x$$

This is an integration in disguise.

$$\lim_{\partial x \to 0} \sum_a^b y\ \partial x = \int_a^b y\ dx$$

Year 2 Pure Chapter 11 – Integration

Which Integration Method?

Non-Trig Functions

- Standard Function
- Reverse Chain Rule
- By Parts or By Substitution
- Partial Fractions / Improper Algebraic Fractions

$f(x)$	How to deal with it	$\int f(x)dx$
$\frac{1}{x}$	Standard function	$\ln x$
$\ln x$	Use IBP, where $u = \ln x, \frac{dv}{dx} = \ln x$	$x\ln x - x$
$\frac{x}{x+1}$	Use algebraic division. $\frac{x}{x+1} \equiv 1 - \frac{1}{x+1}$	$x - \ln\lvert x+1\rvert$
$\frac{1}{x(x+1)}$	Use partial fractions.	$\ln\lvert x\rvert - \ln\lvert x+1\rvert$
$\frac{4x}{x^2+1}$	Reverse chain rule.	$2\ln\lvert x^2+1\rvert$
$\frac{x}{(x^2+1)^2}$	Power around denominator so rewrite as $x(x^2+1)^{-2}$ then reverse chain rule	$-\frac{1}{2}(x^2+1)^{-1}$
e^{2x+1}	Divide by coefficient of x	$\frac{1}{2}e^{2x+1}$
$x\sqrt{2x+1}$	IBP or use sensible substitution. $u = 2x+1$ or even better, $u^2 = 2x+1$.	$\frac{1}{15}(2x+1)^{\frac{3}{2}}(3x-1)$

Trig Functions

- Standard Function
- Formula Sheet
- Reverse Chain Rule
- By Parts or By Substitution
- Rewrite using Trig Identities

$f(x)$	How to deal with it	$\int f(x)dx$
$\sin x$	Standard function	$-\cos x$
$\cos x$	Standard function	$\sin x$
$\tan x$	Formula booklet	$\ln\lvert\sec x\rvert$
$\sin^2 x$	Double Angle for $\cos 2x$ $\cos 2x = 1 - 2\sin^2 x$ $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$	$\frac{1}{2}x - \frac{1}{4}\sin 2x$
$\cos^2 x$	Double Angle for $\cos 2x$ $\cos 2x = 2\cos^2 x - 1$ $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$	$\frac{1}{2}x + \frac{1}{4}\sin 2x$
$\tan^2 x$	$1 + \tan^2 x \equiv \sec^2 x$ $\tan^2 x \equiv \sec^2 x - 1$	$\tan x - x$
$cosec\, x$	Formula booklet	$-\ln\lvert cosec\, x + \cot x\rvert$
$\sec x$	Formula booklet	$\ln\lvert\sec x + \tan x\rvert$
$\cot x$	Formula booklet	$ln\lvert sin\, x\rvert$
$cosec^2 x$	By observation.	$-\cot x$
$\sec^2 x$	Formula booklet	$\tan x$
$\cot^2 x$	$1 + \cot^2 x \equiv cosec^2 x$	$-\cot x - x$
$\sin 2x \cos 2x$	For any product of sin and cos with same coefficient of x, use double angle. $\sin 2x \cos 2x \equiv \frac{1}{2}\sin 4x$	$-\frac{1}{8}\cos 4x$
$\sin^5 x\ cos\, x$	Reverse chain rule.	$\frac{1}{6}sin^6 x$